

Hodge–Gaussian maps ^{*}

Elisabetta Colombo

Dipartimento di Matematica, Università di Milano

Gian Pietro Pirola

Dipartimento di Matematica, Università di Pavia

Alfonso Tortora

Dipartimento di Matematica, Università di Milano

Abstract

Let X be a compact Kähler manifold, and let L be a line bundle on X . Define $I_k(L)$ to be the kernel of the multiplication map $\mathrm{Sym}^k H^0(L) \rightarrow H^0(L^k)$. For all $h \leq k$, we define a map

$$\rho : I_k(L) \rightarrow \mathrm{Hom}(H^{p,q}(L^{-h}), H^{p+1,q-1}(L^{k-h})).$$

When $L = K_X$ is the canonical bundle, the map ρ computes a second fundamental form associated to the deformations of X .

If $X = C$ is a curve, then ρ is a lifting of the Wahl map $I_2(L) \rightarrow H^0(L^2 \otimes K_C^2)$.

We also show how to generalize the construction of ρ to the cases of harmonic bundles and of couples of vector bundles.

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Introduction

In connection with the variations of Hodge structures (VHS), a number of authors have tackled the higher differentials of the period map.

A first definition of second fundamental form (2ff) for a VHS of odd weight is given in [CGGH]. More recently, Karpishpan ([K]) has defined a 2ff for VHS and showed a way to compute it for VHS coming from geometry, using Archimedean cohomology. In the case of curves, he asks whether this 2ff, at any given point, lift $I_2(K_X) \rightarrow H^0(K_X^4)$, the second Wahl (or Gaussian) map for the canonical bundle.

In the projective case, second (and higher) fundamental forms are defined for algebraic varieties, with respect to a fixed projective embedding (cf. [GH2], [L].)

In [G], with reference to unpublished work of Green–Griffiths, it is reported that the projective 2ff (in the sense of [GH2]) of any local Plücker embedding of the moduli space of curves gives, as a quotient, the second Wahl map of the canonical line bundle.

Both kinds of 2ff, for VHS and projective embeddings, can be interpreted as instances of the (classical) 2ff $\Pi := \pi \nabla|_{\mathcal{S}}$ associated to an extension of sheaves $0 \rightarrow \mathcal{S} \rightarrow E \xrightarrow{\pi} \mathcal{Q} \rightarrow 0$, with E a vector bundle with connection ∇ .

In this paper we define a family of maps, that we propose to call *Hodge–Gaussian maps*, existing under very general conditions, namely for line bundles over compact Kähler manifolds. When applied to the canonical bundle, the Hodge–Gaussian map is a 2ff naturally associated to a deformation of the manifold (see theorem 2.1.) If we are dealing with curves, we answer in the affirmative to the question asked in [K], consistently with the statement of [G], cited earlier. Actually, our result holds in a more general setup than those of both [G] and [K], in that it concerns not only the canonical bundle, but any line bundle on a curve. Also, the possibility of making explicit computations, at least in the case of curves, as in lemma 3.2, seems to the authors a step towards understanding the curvature of the moduli space of curves.

The starting point for this paper was a construction of one of the authors (cf. [Pi]), that turned out to be a special case of ours. The hunch that it should be a kind of 2ff, and an attempt at understanding it as a lifting of a Wahl map, in the spirit of Green–Griffiths, lead us to the present results.

The main idea underlying all of our maps is the following:

Let L be a line bundle over a compact Kähler manifold X , with $h^0(L) > 1$. Set $I_2(L) := \ker(\text{Sym}^2 H^0(L) \rightarrow H^0(L^2))$. If $\xi = [\theta] \in H^1(L^{-1})$, θ a Dolbeault representative of ξ , and λ_i , $i = 1, \dots, r$, is a basis of $H^0(L)$, then the cup products $\theta\lambda_i \in \mathcal{A}^{0,1}(X)$ have harmonic decompositions $\theta\lambda_i = \gamma_i + \bar{\partial}h_i$. Now, for any $Q = \sum a_{ij}\lambda_i \otimes \lambda_j \in I_2(L)$, the section

$$\sum a_{ij}\lambda_i \partial h_j \in A^{1,0}(L)$$

determines an element of $H^0(L \otimes \Omega_X^1)$.

It turns out that the map $I_2(L) \otimes H^1(L^{-1}) \rightarrow H^0(L \otimes \Omega_X)$ is well defined. Especially, when X is a curve, this map, seen as a map $I_2(L) \rightarrow H^0(L \otimes K_X) \otimes H^0(L \otimes K_X)$, is a lifting of the second Wahl map for L , $\mu_2 : I_2(L) \rightarrow H^0(L^2 \otimes K_X^2)$, with respect to the natural multiplication map $H^0(L \otimes K_X) \otimes H^0(L \otimes K_X) \rightarrow H^0(L^2 \otimes K_X^2)$.

The crux of our construction is the harmonic decomposition of the (p,q)-forms, to define the map, and the principle of two types, to prove that it is well-defined.

This observation allows us to generalize the construction to a map

$$I_k(L) \otimes H^{p,q}(L^{-h}) \rightarrow H^{p+1,q-1}(L^{k-h}),$$

defined for line bundles L on X .

Actually, the basic trick in the definition of the map is a switch from $\bar{\partial}$ to ∂ , and it works also in more general situations, provided some kind of harmonic decomposition exist, for which the principle of two types holds. This is the case for harmonic bundles, which admit the same kind of maps. Such a generalization is not gratuitous, but with an eye towards finding interactions between Hodge theory and the equations defining an algebraic variety.

The authors' opinion is that the main interest of the present paper resides in the construction of a natural map ρ , not hiterto known in the literature. Indeed, in the published account [G] of the work of Green–Griffiths cited above, there is no mention of it.

Several people, whose encouragement we gratefully acknowledge, held the opinion that the non-holomorphic map ρ could be a suitable projection of an algebraic one. Its being non-holomorphic is likely to be the main obstruction to a more systematic use of ρ in algebraic geometry. However, a most likely application of ρ should be found in the investigation of the curvature properties of certain moduli spaces, which fact nicely ties in with the

non-holomorphicity. On the other hand, also the 2ff defined in [K] is non-holomorphic even though it is somewhat shrouded in the use of Archimedean cohomology.

The paper is organized as follows:

In section 1 we define the Hodge–Gaussian map

$$\rho : I_k(L) \otimes H^{p,q}(L^{-h}) \rightarrow H^{p+1,q-1}(L^{k-h})$$

whose construction is outlined above. We also note some formal properties of the map, which are summarized in proposition 1.9.

In section 2 we compare our map and the 2ff. Given a smooth deformation $\mathcal{X} \xrightarrow{\psi} B$ of $X = X_{b_0}$, let $K_{\mathcal{X}|B}$ be the relative canonical bundle. We show that the 2ff associated to the map $\mathrm{Sym}^k \psi_* K_{\mathcal{X}|B} \rightarrow \psi_* K_{\mathcal{X}|B}^k$, at the point $b_0 \in B$, is factorized by $I_k(K_X) \otimes H^{n-1,1}(K_X^{-1}) \xrightarrow{\rho} H^{n,0}(K_X^{k-1})$, through the Kodaira–Spencer map $\kappa : T_{B,b_0} \rightarrow H^1(T_X) \simeq H^{n-1,1}(K_X^{-1})$.

Section 3 deals with the case when $X = C$ is a curve: we show that ρ gives a lifting of the Wahl map.

In section 4 we show how to carry the construction of ρ over to more general situations, defining a Hodge–Gaussian map also in the following cases: (a) for couples of vector bundles E, F —with $I_2(L)$ replaced by the second module of relations $R_2(E, F)$ —and (b) for harmonic bundles.

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1 The main construction

Let X be a compact Kähler manifold, $\dim X = n$, and let L be a line bundle over X , $h^0(L) = r > 0$.

The goal of this section is to define the *Hodge–Gaussian map*

$$\rho : I_k(L) \rightarrow \mathrm{Hom}(H^{p,q}(L^{-m}), H^{p+1,q-1}(L^{k-m})),$$

where $I_k(L) := \ker(\mathbf{m}_k : \mathrm{Sym}^k H^0(L) \rightarrow H^0(L^k))$, \mathbf{m}_k being the multiplication map.

To do so, we need the following classical results of Hodge theory (see e.g.

[GH1], p. 84 and 149; also, for a thorough exploitation of the principle of two types, [DGMS].)

Theorem 1.1 *Let X be a compact Kähler manifold.*

1. (Hodge theorem) *Any $\bar{\partial}$ -closed form $\alpha \in A^{p,q}(X)$ has a unique harmonic representative, hence can be written as $\alpha = \gamma + \bar{\partial}h$, with $\gamma \in \mathcal{H}^{p,q}$ harmonic and $h \in A^{p,q-1}(X)$.*

2. (Principle of two types) *Let $\alpha \in A^{p,q}(X)$ satisfy $\partial\alpha = \bar{\partial}\alpha = 0$ and be either ∂ - or $\bar{\partial}$ -exact. Then for some $\beta \in A^{p-1,q-1}(X)$, $\alpha = \partial\bar{\partial}\beta$.*

We now introduce some multi-index notation.

Fix a basis $\lambda_1, \dots, \lambda_r$ of $H^0(L)$.

Define $R_k := \{1, 2, \dots, r\}^k$. If $S = (s_1, \dots, s_h) \in R_h$ and $T = (t_1, \dots, t_k) \in R_k$, we denote $ST := (s_1, \dots, s_h, t_1, \dots, t_k) \in R_{h+k}$.

For any $J \in R_k$ we write: $a_J = a_{j_1 \dots j_k} \in \mathbb{C}$ is a scalar,

$$\lambda_{\otimes J} = \lambda_{j_1} \otimes \dots \otimes \lambda_{j_k} \in \otimes^k H^0(L),$$

$$\lambda_J = \lambda_{j_1} \dots \lambda_{j_k} \in H^0(L^k).$$

Clearly, an element $P \in I_k(L)$ is uniquely written as $\sum_{J \in R_k} a_J \lambda_{\otimes J}$, with the a_J 's symmetric in the j 's, satisfying $\sum_{J \in R_k} a_J \lambda_J = 0$.

In standard multi-index notation, $P \in I_k(L)$ can be thought of as a polynomial of degree k , $\sum_{|K|=k} a_K x^K$, vanishing in λ , i.e. $P(\lambda) = \sum_{|K|=k} a_K \lambda^K = 0$, where $\lambda^K = \lambda_1^{k_1} \dots \lambda_r^{k_r}$.

Proposition–Definition 1.2 : Hodge–Gaussian maps

Given $\xi \in H^{p,q}(L^{-m})$, choose a Dolbeault representative $\theta \in A^{p,q}(L^{-m})$. For any $T \in R_m$, the cup product $\theta\lambda_T \in A^{p,q}(X)$ is $\bar{\partial}$ -closed, so it has a harmonic decomposition

$$\theta\lambda_T = \gamma_T + \bar{\partial}h_T \tag{1}$$

with $\gamma_T \in \mathcal{H}^{p,q}$ and $h_T \in A^{p,q-1}(X)$. Let $P = \sum_{J \in R_k} a_J \lambda_{\otimes J} \in I_k(L)$.

For all $0 < m \leq k$, the following map is well-defined and \mathbb{C} -linear

$$\begin{aligned} \rho : I_k(L) &\rightarrow \text{Hom}(H^{p,q}(L^{-m}), H^{p+1,q-1}(L^{k-m})) \\ P &\rightarrow (\xi \mapsto \rho_P(\xi)) \end{aligned}$$

where $\rho_P(\xi)$ is the Dolbeault cohomology class of the L^{k-m} -valued $(p+1, q-1)$ -form

$$\sigma_P(\theta) := \sum_{\substack{S \in R_{k-m} \\ T \in R_m}} a_{ST} \lambda_S \partial h_T. \tag{2}$$

Proof: We need to check that $\sigma_P(\theta)$ is $\bar{\partial}$ -closed and that $\rho_P(\xi) = [\sigma_P(\theta)]$, as an element of $H^{p+1,q-1}(L^{k-m})$, is independent of the choices made.

(i) $\sigma_P(\theta)$ is $\bar{\partial}$ -closed.

Indeed,

$$\begin{aligned}\bar{\partial}\sigma_P(\theta) &= \sum a_{sT}\lambda_s\bar{\partial}\partial h_T = -\sum a_{sT}\lambda_s\partial(\theta\lambda_T - \gamma_T) \\ &= -\sum a_{sT}\lambda_s\partial(\theta\lambda_T)\end{aligned}$$

because γ_T is harmonic. A local computation shows that $\sum a_{sT}\lambda_s\partial(\theta\lambda_T)$ vanishes: for any $p \in X$, let ℓ and ℓ^* be a local generator of L and its dual in a neighborhood U of p , then $\lambda_i = \phi_i \cdot \ell$, $\theta = \tau \cdot (\ell^*)^m$, with ϕ_i functions and τ a (p,q) -form on U respectively. On U , we have $\lambda_T = \phi_T \ell^m$, where $\phi_T = \phi_{t_1} \cdots \phi_{t_m}$ is a function defined on U , so $\partial(\theta\lambda_T) = \phi_T \partial\tau + (-1)^{p+q}\tau \wedge \partial\phi_T$, hence

$$\sum a_{sT}\lambda_s\partial(\theta\lambda_T) = \left(\partial\tau \sum a_{sT}\phi_s\phi_T + (-1)^{p+q}\tau \wedge \sum a_{sT}\phi_s\partial\phi_T\right) \ell^{k-m} = 0.$$

Indeed, $\sum a_{sT}\lambda_s\lambda_T = 0$ means that the function $\sum a_{sT}\phi_s\phi_T$ is identically zero on U , thus also $\partial(\sum a_{sT}\phi_s\phi_T) = 0$; since the scalars a_j are symmetric with respect to the indices j 's, it is easy to see that $\sum a_{sT}\phi_s\partial\phi_T = \frac{m}{k}\partial(\sum a_{sT}\phi_s\phi_T) = 0$. (see *infra*, remark 1.3)

(ii) $\rho_P(\xi)$ does not depend on the choice of θ .

Let $\tilde{\theta}$ be another Dolbeault representative of ξ , we have $\tilde{\theta} = \theta + \bar{\partial}\chi$, with $\chi \in A^{p,q-1}(L^{-m})$. Now,

$$\tilde{\theta}\lambda_T = \gamma_T + \bar{\partial}\tilde{h}_T,$$

with γ_T unchanged because it is the unique harmonic representative of $\xi\lambda_T \in H^{p,q}(X)$, so

$$\gamma_T + \bar{\partial}\tilde{h}_T = \tilde{\theta}\lambda_T = (\theta + \bar{\partial}\chi)\lambda_T = \theta\lambda_T + \bar{\partial}(\chi\lambda_T) = \gamma_T + \bar{\partial}h_T + \bar{\partial}(\chi\lambda_T),$$

hence

$$\bar{\partial}\tilde{h}_T = \bar{\partial}(h_T + \chi\lambda_T)$$

and

$$\tilde{h}_T = h_T + \chi\lambda_T + g_T,$$

with g_T a $\bar{\partial}$ -closed $(p, q-1)$ -form.

It follows

$$\sum a_{sT}\lambda_s\bar{\partial}\tilde{h}_T = \sum a_{sT}\lambda_s\bar{\partial}h_T + \sum a_{sT}\lambda_s\bar{\partial}g_T,$$

because $\sum a_{sT} \lambda_s \partial(\chi \lambda_T) = 0$ as above. So we need to show that $\sum a_{sT} \lambda_s \partial g_T$ is $\bar{\partial}$ -exact.

As ∂g_T is $\bar{\partial}$ -closed, by the principle of two types $\partial g_T = \bar{\partial} \partial k_T$, hence

$$\sum a_{sT} \lambda_s \partial g_T = \sum a_{sT} \lambda_s \bar{\partial} \partial k_T = \bar{\partial} \left(\sum a_{sT} \lambda_s \partial k_T \right).$$

The linearity of ρ_P and its independence of the choice of a basis of $H^0(L)$ are clear. ■

Remark 1.3 (i) $\rho_P(\xi)$ can also be defined, perhaps more intuitively, thinking of $P \in I_k(L)$ as a polynomial of degree k vanishing in λ , $P(\lambda) = 0$. If we write $P = P(x)$ in the form $\sum_{J \in R_k} a_J x_J$, where a_J 's are the same scalar seen above and $x_J = x_{j_1} \dots x_{j_k}$, then it is easy to see that the partial derivatives of $P(x)$ are given by $\frac{\partial P}{\partial x_i} = k \sum_{S \in R_{k-1}} a_{Si} x_S$. Also, when $\xi \in H^{p,q}(L^{-1})$, (1) becomes $\theta \lambda_i = \gamma_i + \bar{\partial} h_i$, for all $i = 1, \dots, r$. Thus $\rho_P(\xi)$ is the cohomology class of the form

$$\sigma_P(\theta) = \frac{1}{k} \sum_{i=1}^r \frac{\partial P}{\partial x_i}(\lambda) \partial h_i.$$

For $m > 1$, the formula expressing $\rho_P(\xi)$ in terms of higher-order derivatives of $P(x)$ is slightly more complicated. For all $T = (t_1, \dots, t_m) \in R_m$, let $\partial_T P = \frac{\partial^m P}{\partial x_{t_1} \dots \partial x_{t_m}}$, then one sees that $\partial_T P = \frac{k!}{(k-m)!} \sum_{S \in R_{k-m}} a_{ST} x_S$, so $\sigma_P(\theta) = \sum_{S,T} a_{ST} \lambda_S \partial h_T = \frac{(k-m)!}{k!} \sum_T \partial_T P(\lambda) \partial h_T$. Now, in standard multi-index notation, $\partial_T P = \frac{\partial^m P}{\partial x^\tau}$, where $\tau = \tau(T) = (\tau_1, \dots, \tau_r)$, τ_j being how many times j appears in $T = (t_1, \dots, t_m)$. Also, the same derivative $\frac{\partial^m P}{\partial x^\tau}$ is repeated $\frac{m!}{\tau_1! \dots \tau_r!}$ times, corresponding to the different $T \in R_m$ which give the same $\tau(T)$. Summing up, we obtain

$$\sigma_P(\theta) = \frac{m!(k-m)!}{k!} \sum_{|I|=m} \frac{1}{i_1! \dots i_r!} \frac{\partial^m P}{\partial x^I}(\lambda) \partial h^I,$$

with h^I given by the decomposition (1) relative to $\theta \lambda^I = \theta \lambda_1^{i_1} \dots \lambda_r^{i_r}$.

(ii) The basic trick in the definition of ρ_P is to take the $\bar{\partial}$ -exact part of the decomposition of a form and then switch to a ∂ -exact form, i.e. going from $\bar{\partial} h_T$ to ∂h_T . To do so, we just need the two facts of theorem 1.1, hence a similar construction can be carried out also in other more general situations,

where we have some kind of harmonic decomposition, for which the principle of two types holds.

Proposition 1.4 *If X is a compact complex manifold having several Kähler metrics compatible with its complex structure, then the map ρ_P is independent of the (Kähler) metric used to define it, and is completely determined by the underlying complex structure of X .*

Proof: Let K_1 and K_2 be the harmonic projectors coming from two different Kähler metrics on X ; then, for any $\bar{\partial}$ -closed form ω we have the harmonic decompositions $\omega = K_i\omega + \bar{\partial}h_i$, $i = 1, 2$. Set $\psi := \partial(h_1 - h_2) = \partial h$.

We claim that ψ is $\bar{\partial}$ -exact.

Indeed, $\bar{\partial}h$ is ∂ -closed, (because $\bar{\partial}h = K_2\omega - K_1\omega$, with the $K_i\omega$ harmonic forms) so the principle of two types implies that $\bar{\partial}h = -\bar{\partial}\partial f$, or, equivalently, $\bar{\partial}(h + \partial f) = 0$. Therefore, $h + \partial f$ has harmonic decomposition $h + \partial f = \tau + \bar{\partial}l$. It follows that $\psi = \partial h = \partial(h + \partial f) = \partial\tau + \partial\bar{\partial}l = \bar{\partial}(-\partial l)$.

Going back to our situation, $\theta\lambda_T$ has harmonic decompositions, with respect to the different Kähler structures, $\theta\lambda_T = \gamma_T + \bar{\partial}h_T = \delta_T + \bar{\partial}g_T$, hence $\partial(h_T - g_T) = \bar{\partial}l_T$ is $\bar{\partial}$ -exact. It follows that

$$\sum a_{sT}\lambda_s\partial h_T - \sum a_{sT}\lambda_s\partial g_T = \sum a_{sT}\lambda_s\bar{\partial}l_T = \bar{\partial}\left(\sum a_{sT}\lambda_sl_T\right),$$

thus the cohomology classes

$$\left[\sum a_{sT}\lambda_s\partial h_T\right] = \left[\sum a_{sT}\lambda_s\partial g_T\right]$$

are equal in $H^{p+1,q-1}(L^{k-m})$. ■

Remark 1.5 ρ_P does not vary holomorphically on family of varieties, in the following sense.

Let $\mathcal{X} \rightarrow S$ be a smooth analytic family of Kähler manifolds and let $\mathcal{L} \rightarrow \mathcal{X}$ be a line bundle. Define $\mathcal{H}^{p,q}(\mathcal{L}^k) := R^q\pi_*(\Omega_{\mathcal{X}|S}^p \otimes L^k)$ and $\mathcal{I}_r(\mathcal{L}) := \ker(\text{Sym}^r\pi_*\mathcal{L} \rightarrow \pi_*\mathcal{L}^r)$. ρ extends to a map

$$\tilde{\rho} : \mathcal{I}_k(\mathcal{L}) \otimes \mathcal{H}^{p,q}(\mathcal{L}^{-m}) \rightarrow \mathcal{H}^{p+1,q-1}(\mathcal{L}^{k-m})$$

which is not holomorphic, but only real-analytic.

The maps ρ have a few more properties worth noting.

Proposition 1.6 *For all $\alpha \in H^{s,t}(X)$ and $\xi \in H^{p,q}(L^{-m})$,*

$$\rho_P(\alpha \cdot \xi) = \alpha \cdot \rho_P(\xi).$$

Proof: Choose a harmonic representative β of the class α , then, recalling the notation of (1), $\beta \wedge \gamma_T$, which is not harmonic, has a harmonic decomposition

$$\beta \wedge \gamma_T = \sigma_T + \bar{\partial}g_T.$$

The forms β, γ_T and σ_T , being harmonic, are both ∂ - and $\bar{\partial}$ -closed, hence $\bar{\partial}g_T$ is ∂ -closed, so, by the principle of two types, $\bar{\partial}g_T = \bar{\partial}\partial k_T$, for a suitable k_T . It follows that $\beta \wedge \theta\lambda_T = \beta \wedge (\gamma_T + \bar{\partial}h_T) = \beta \wedge \gamma_T + (-1)^{s+t}\bar{\partial}(\beta \wedge h_T) = \sigma_T + \bar{\partial}\partial k_T + (-1)^{s+t}\bar{\partial}(\beta \wedge h_T) = \sigma_T + \bar{\partial}(\partial k_T + (-1)^{s+t}\beta \wedge h_T)$, so we have the harmonic decomposition

$$\beta \wedge \theta\lambda_T = \sigma_T + \bar{\partial}f_T,$$

with $f_T = \partial k_T + (-1)^{s+t}\beta \wedge h_T$, hence $\partial f_T = \beta \wedge \partial h_T$ —recall that β is ∂ -closed. The outcome is that

$$\begin{aligned} \rho_P(\alpha \cdot \xi) &= \left[\sum a_{sT} \lambda_s \partial f_T \right] = \left[\sum a_{sT} \lambda_s \beta \wedge \partial h_T \right] \\ &= \left[\beta \wedge \sum a_{sT} \lambda_s \partial h_T \right] = [\beta] \cdot \left[\sum a_{sT} \lambda_s \partial h_T \right] = \\ &= \alpha \cdot \rho_P(\xi) \quad \blacksquare \end{aligned}$$

Proposition 1.7 *For all $P \in I_k(L)$, $\xi \in H^{p,q}(L^{-m})$ and $\eta \in H^{n-p-1, n-q+1}(L^{-k+m})$,*

$$\xi \rho_P(\eta) = (-1)^{p+q+1} \eta \rho_P(\xi).$$

Proof: Let $\theta \in A^{p,q}(L^{-m})$ and $\chi \in A^{n-p-1, n-q+1}(L^{-k+m})$ be Dolbeault representatives of ξ and η respectively. Given $S \in R_{k-m}, T \in R_m$, consider the corresponding harmonic decompositions $\theta\lambda_T = \gamma_T + \bar{\partial}h_T$ and $\chi\lambda_S = \delta_S + \bar{\partial}k_S$. Then the cohomology class $\eta\rho_P(\xi) \in H^{n,n}(X)$ has Dolbeault representative

$$\chi \cdot \sum a_{sT} \lambda_s \partial h_T = \sum a_{sT} \chi \lambda_s \wedge \partial h_T = \sum a_{sT} \delta_s \wedge \partial h_T + \sum a_{sT} \bar{\partial}k_s \wedge \partial h_T.$$

It is easy to see that $d(\delta_s \wedge h_T) = \delta_s \wedge \partial h_T$, hence $\eta\rho_P(\xi)$ is represented also by the form $\sum a_{sT} \bar{\partial} k_s \wedge \partial h_T$. Similarly, $\xi\rho_P(\eta)$ is represented by $\sum a_{Ts} \bar{\partial} h_T \wedge \partial k_s$. Now, if $h \in A^{p,q-1}(X)$ and $k \in A^{n-p-1,n-q}(X)$, it is true in general that $[\bar{\partial} h \wedge \partial k] = (-1)^{p+q+1} [\bar{\partial} k \wedge \partial h]$ in $H^{n,n}(X)$.

Indeed, taking into account the number of dz 's and $d\bar{z}$'s, one sees that $d(h \wedge dk) = \partial h \wedge \bar{\partial} k + \bar{\partial} h \wedge \partial k$, thus $[\bar{\partial} h \wedge \partial k] = -[\partial h \wedge \bar{\partial} k]$ in $H^{n,n}(X)$. Since $\partial h \wedge \bar{\partial} k = (-1)^{(p+q)(2n-p-q)} \bar{\partial} k \wedge \partial h = (-1)^{p+q} \bar{\partial} k \wedge \partial h$, then $[\bar{\partial} h \wedge \partial k] = (-1)^{p+q+1} [\bar{\partial} k \wedge \partial h]$.

The conclusion is now clear—recall that the a_j 's are symmetric with respect to the indices j 's:

$$\begin{aligned} \xi\rho_P(\eta) &= \left[\sum a_{Ts} \bar{\partial} h_T \wedge \partial k_s \right] = (-1)^{p+q+1} \left[\sum a_{sT} \bar{\partial} k_s \wedge \partial h_T \right] \\ &= (-1)^{p+q+1} \eta\rho_P(\xi) \quad \blacksquare \end{aligned}$$

Notation 1.8 *Given a line bundle L as before, write*

$$H^\bullet(L^\bullet) := \oplus_{p,q,k} H^{p,q}(L^k),$$

with $p, q, k \in \mathbb{N}_0$, $p, q \leq n$.

Furthermore, standard notations are

$$H^\bullet(X) := \oplus_{p,q} H^{p,q}(X) \quad \text{and} \quad I(L) := \oplus_k I_k(L).$$

Clearly, $H^\bullet(L^\bullet)$ has a structure of $H^\bullet(X)$ –module, given by the cup product. Using the identification $H^{p,q}(L^{-m})^* = H^{n-p,n-q}(L^m)$, the map $\rho_P \in \text{Hom}(H^{p,q}(L^{-m}), H^{p+1,q-1}(L^{k-m}))$ is an element of $H^{n-p,n-q}(L^m) \otimes_{\mathbb{C}} H^{p+1,q-1}(L^{k-m})$, hence ρ is a map

$$\rho : I_k(L) \rightarrow H^{n-p,n-q}(L^m) \otimes_{\mathbb{C}} H^{p+1,q-1}(L^{k-m}).$$

Note that, when L is ample, ρ is nonzero only when $p + q = n$.

Putting the ρ 's together, for all values of k , and taking into account the linearity expressed in proposition 1.6, we have a map

$$\rho : I(L) \rightarrow H^\bullet(L^\bullet) \otimes_{H^\bullet(X)} H^\bullet(L^\bullet).$$

Thinking of $I(L)$ and $H^\bullet(L^\bullet) \otimes_{H^\bullet(X)} H^\bullet(L^\bullet)$ as graded \mathbb{C} -modules, via

$$I(L) = \oplus I_k(L), \quad H^\bullet(L^\bullet) \otimes_{H^\bullet(X)} H^\bullet(L^\bullet) = \oplus_k \left(\oplus_{m+j=k} H^\bullet(L^m) \otimes_{H^\bullet(X)} H^\bullet(L^j) \right),$$

ρ is then a map of graded modules.

Proposition 1.7 expresses the fact that $\rho(P) \in H^\bullet(L^\bullet) \otimes_{H^\bullet(X)} H^\bullet(L^\bullet)$ is invariant with respect to the involution

$$\iota(\xi \otimes \eta) := (-1)^{\deg \xi \cdot \deg \eta + 1} \eta \otimes \xi,$$

where $\deg \xi = p + q$ for $\xi \in H^{p,q}(L^k)$.

We can summarize the remarks above in the following

Proposition 1.9 $\rho : I(L) \rightarrow H^\bullet(L^\bullet) \otimes_{H^\bullet(X)} H^\bullet(L^\bullet)$ is a map of graded \mathbb{C} -modules. Its image is contained in the subspace of $H^\bullet(L^\bullet) \otimes_{H^\bullet(X)} H^\bullet(L^\bullet)$ invariant with respect to the involution $\iota(\xi \otimes \eta) = (-1)^{\deg \xi \cdot \deg \eta + 1} \eta \otimes \xi$. ■

Especially, when $n = \dim X$ is odd, $n = 2m + 1$, the map ρ is symmetric on the middle cohomology, i.e.

$$\rho : I_{2k}(L) \rightarrow \text{Sym}^2 H^{m+1,m}(L^k).$$

2 Hodge–Gaussian map and second fundamental form

Let X be a complex manifold and let E be a holomorphic vector bundle on X , with connection $\nabla : \mathcal{A}^0(E) \rightarrow \mathcal{A}^1(E)$. For any exact sequence of sheaves of \mathcal{O}_X -modules $0 \rightarrow \mathcal{S} \rightarrow E \xrightarrow{\pi} \mathcal{Q} \rightarrow 0$, the second fundamental form (2ff) of \mathcal{S} in E is the $\mathcal{A}^0(X)$ -linear map

$$\Pi : \mathcal{A}^0(\mathcal{S}) \rightarrow \mathcal{A}^1(\mathcal{Q})$$

defined by $\Pi(\sigma) := \pi \nabla|_{\mathcal{S}}(\sigma)$. If ∇ is compatible with the complex structure, then Π lands into $\mathcal{A}^{1,0}(\mathcal{Q})$, hence $\Pi \in \mathcal{A}^{1,0}(\text{Hom}(\mathcal{S}, \mathcal{Q}))$ (see e.g. [GH1].) With an eye on the case at hand, we slightly enlarge the definition of 2ff by allowing an exact sequence of type $0 \rightarrow \mathcal{S} \rightarrow E \xrightarrow{\pi} \mathcal{Q}$, for which π is not necessarily surjective; clearly, the same definition of Π makes still sense.

We are most interested in the following situation:

Let $\mathcal{X} \xrightarrow{\psi} B$ be a smooth analytic family of compact Kähler manifolds of dimension n , i.e. assume that ψ is a submersion, so all fibers are smooth. For all k , there are exact sequences

$$0 \rightarrow \mathcal{I}_k(K_{\mathcal{X}|B}) \rightarrow \mathrm{Sym}^k(\psi_* K_{\mathcal{X}|B}) \xrightarrow{\mathbf{m}} \psi_* K_{\mathcal{X}|B}^k, \quad (3)$$

whose 2ff we denote by Π_k . Here $K_{\mathcal{X}|B}$ is the relative canonical bundle, $K_{\mathcal{X}|B} = \wedge^n \Omega_{\mathcal{X}|B}^1$, $\Omega_{\mathcal{X}|B}^1$ being the relative cotangent bundle. Also, \mathbf{m} is the natural multiplication map.

Recall that the fiber of $\psi_* K_{\mathcal{X}|B}$ on the point $b \in B$ is $H^0(X_b, K_{X_b})$, so $\psi_* K_{\mathcal{X}|B} = \mathcal{F}^n \subseteq \mathbf{R}^n \psi_* \mathbb{C}$ is a piece of the Hodge filtration and, as such, has a natural metric connection induced by ∇^{GM} , the flat Gauss–Manin (GM) connection on the polarized VHS $\mathbf{R}^n \psi_* \mathbb{C}$. The 2ff of the exact sequence (3)

$$\Pi_k : \mathcal{I}_k \rightarrow \psi_* K_{\mathcal{X}|B}^k \otimes \Omega_B,$$

becomes, on the central fiber,

$$\Pi_k : I_k \otimes T_{B,b_0} \rightarrow H^0(K_X^k).$$

Now, the Kodaira–Spencer (KS) map of the family \mathcal{X} is (on the central fiber)

$$\kappa : T_{B,b_0} \rightarrow H^1(T_X).$$

With the identifications $H^0(K_X^k) = H^0(K_X^{k-1} \otimes \Omega_X^n) = H^{n,0}(K_X^{k-1})$ and $H^1(T_X) = H^1(\Omega_X^*) = H^1(\Omega_X^{n-1} \otimes K_X^{-1}) = H^{n-1,1}(K_X^{-1})$, we have the following statement.

Theorem 2.1 *The diagram*

$$\begin{array}{ccc} I_k \otimes T_{B,b_0} & \xrightarrow{id \otimes \kappa} & I_k \otimes H^{n-1,1}(K_X^{-1}) \\ \Pi_k \downarrow & & \downarrow \rho \\ H^{n,0}(K_X^{k-1}) & \xlongequal{\quad} & H^{n,0}(K_X^{k-1}) \end{array}$$

is commutative up to a constant.

The strategy of proof is very simple. First, it is enough to consider only the case of one-dimensional deformations $\mathcal{X} \rightarrow \Delta$, hence we need to check the equality for just one vector $\mathbf{v} \in T_{\Delta,0}$. We now compute both the 2ff and

the KS map using a fixed C^∞ -trivialization $\mathcal{X} \simeq \Delta \times X$. Finally, for any given $P \in I_k$, we plug in the value $\kappa(\mathbf{v})$ in the expression of ρ_P and get $\Pi_{k,\mathbf{v}}(P) = \rho_P(\kappa(\mathbf{v}))$, up to a constant.

Of course, there is a slight abuse of notation in denoting with ρ the map of the present theorem, map which is actually ρ only after an obvious duality.

Proof: Since the statement is local, we can suppose that \mathcal{X} be a one-dimensional deformation, i.e. $\mathcal{X} \xrightarrow{\psi} \Delta$ be parameterized by the unit circle $\Delta = \{|t| < 1\}$, with $X_0 = X$ and $\mathbf{v} = \frac{\partial}{\partial t} \in T_{\Delta,0}$. The Hodge bundle $\mathcal{H}_\mathbb{C}^n = \mathbf{R}^n \psi_* \mathbb{C}$ is a flat bundle, with a flat connection ∇^{GM} , the (local) GM connection. The GM connection induces the connection $\nabla^{n,0}$ on the subbundle $\mathcal{H}^{n,0} = F^n \mathcal{H}_\mathbb{C}^n = \psi_* K_{\mathcal{X}|\Delta}$, which in turn induces a connection, denoted by ∇ , on the symmetric product $\text{Sym}^k \mathcal{H}^{n,0}$.

We now compute both the GM connection and the KS map following the method set forth in [G] pp. 30–32, which we briefly summarize here.

Let Y be a C^∞ -lifting of the holomorphic vector field $\frac{\partial}{\partial t}$ on Δ ; then we get a C^∞ -trivialization $\Delta \times X \xrightarrow{\tau} \mathcal{X}$ by $\tau(x, t) := \Phi_{tY}(1)$, where $\Phi_Y(t)$ denotes the flow associated to the vector field Y .

One sees that $\bar{\partial}Y|_X \in A^{0,1}(T_{\mathcal{X}}|_X)$ is actually a closed form $\theta = \bar{\partial}Y|_X$ in $A^{0,1}(T_X)$ that represents the KS class associated to $\frac{\partial}{\partial t}$, i.e. $\kappa(\frac{\partial}{\partial t}) = [\bar{\partial}Y|_X] = [\theta]$.

Let $\omega(t)$ be a section of $\mathcal{H}^{n,0}$, then, for all t , $\omega(t) \in H^0(K_{X_t})$; we may think of $\omega(t)$ as $\Omega \in A^{n,0}(\mathcal{X})$ such that $\Omega|_{X_t} = \omega(t)$ as $(n, 0)$ -forms on X_t .

The isomorphism $\tau_t : X \rightarrow X_t$, induced by τ , gives an inclusion $\tau_t^* : A^{n,0}(X_t) \hookrightarrow A^n(X)$. Since $\omega(t) \in A^{n,0}(X_t)$ is d -closed, so is also $\tau_t^*(\omega(t)) \in A^n(X)$, thus we obtain a power series expansion around $t = 0$

$$\tau_t^*(\omega(t)) = \omega + (\alpha + dh)t + o(t^2),$$

with $\omega = \omega(0)$, α a harmonic n -form and h an $(n-1)$ -form.

It follows that, as cohomology classes, $\nabla_{\frac{\partial}{\partial t}}^{\text{GM}}[\omega(t)]_{t=0} = [\alpha]$.

On the other hand, $\frac{\partial \omega(t)}{\partial t} = \tau^* \mathcal{L}_Y \Omega = \tau^* \langle d\Omega, Y \rangle + \tau^* d \langle \Omega, Y \rangle$ as forms, so $\frac{\partial \omega(t)}{\partial t}$ has at least $n-1$ dz 's, hence $\alpha + dh$ lives in $A^{n,0}(X) \oplus A^{n-1,1}(X)$ and is of type $\alpha = \alpha^{n,0} + \alpha^{n-1,1}$, $h \in A^{n-1,0}(X)$.

Finally, the $(n-1, 1)$ part of $\frac{\partial \omega(t)}{\partial t}|_{t=0}$ is the contraction of Ω with $\bar{\partial}Y$, restricted to X . So we have the harmonic decomposition

$$\theta\omega = \alpha^{n-1,1} + \bar{\partial}h, \tag{4}$$

and $\kappa(\frac{\partial}{\partial t}) \cdot \omega = [\alpha^{n-1,1}]$.

Now the conclusion of the proof is straightforward.

Let $P \in I_k$, it is of type $P = \sum a_J \omega_{\otimes J}$, so $\Pi_{k, \frac{\partial}{\partial t}}(P) = \mathbf{m}(\nabla_{\frac{\partial}{\partial t}} \sigma|_{t=0})$, where $\sigma(t)$ is a section through $P = \sigma(0)$. Since $\nabla_{\frac{\partial}{\partial t}}^{n,0} \omega_j(t)|_{t=0} = \alpha_j^{n,0}$, we see that $\Pi_{k, \frac{\partial}{\partial t}}(P)$ is represented by the form $\sum \dot{a}_J(0) \omega_J + k \sum_{L \in R_{k-1}}^{j=1 \dots g} a_{Lj}(0) \omega_L \alpha_j^{n,0}$ — here $g = h^0(K_X)$; recall that $R_{k-1} = \{1, \dots, g\}^{k-1}$, also see remark 1.3. Since $\sigma(t)$ is a section of \mathcal{I}_k , we have that $\sum a_J(t) \omega(t)_J = 0$ identically, so also its derivative with respect to t vanishes at $t = 0$, i.e. $\sum \dot{a}_J(0) \omega_J + k \sum a_{Lj}(0) \omega_L (\alpha_j + dh_j) = 0$, and, taking the $(n, 0)$ part of $\alpha_j + dh_j$, $\sum \dot{a}_J(0) \omega_J + k \sum a_{Lj}(0) \omega_L \alpha_j^{n,0} = -k \sum a_{Lj}(0) \omega_L \partial h_j$. In other words, up to a constant factor,

$$\Pi_{k, \frac{\partial}{\partial t}}(P) = \sum a_{Lj} \omega_L \partial h_j,$$

where $a_{Lj} = a_{Lj}(0)$.

To compute $\rho_P(\kappa(\frac{\partial}{\partial t}))$ we take θ as representative of $\kappa(\frac{\partial}{\partial t})$, so we have the harmonic decompositions (4) relative to the products $\theta \omega_j$, i.e. $\theta \omega_j = \alpha_j^{n-1,1} + \bar{\partial} h_j$, hence

$$\rho_P(\kappa(\frac{\partial}{\partial t})) = \sum a_{Lj} \omega_L \partial h_j,$$

and the theorem is proved. ■

Remark 2.2 We can also interpret ρ as follows.

Fix $\xi \in H^1(T_X)$, then there is a map

$$\begin{array}{ccc} \tau_\xi : \oplus_k I_k(K_X) & \rightarrow & \oplus_k H^0(K_X^k) \\ P & \rightarrow & \rho_P(\xi) \end{array}$$

Suppose that the canonical map $\iota_{K_X} : X \rightarrow \mathbb{P} H^0 K_X^* = \mathbb{P}^n$ is an embedding, then the polynomials in this \mathbb{P}^n are $\text{Sym} H^0(K_X) = \mathbb{C}[\mathbf{x}]$ and the ideal of the image $X \simeq \iota_{K_X}(X) \subseteq \mathbb{P}^n$ is $I = \oplus_k I_k(K_X)$; furthermore, if it is projectively normal, then its homogeneous coordinate ring $S = \frac{\mathbb{C}[\mathbf{x}]}{I}$ coincides with $\oplus_k H^0(K_X^k)$. It is easily seen that τ_ξ is $\mathbb{C}[\mathbf{x}]$ -linear, so it factors through the quotient, $\tau_\xi : I/I^2 \rightarrow S$.

Summing up, we have a linear map of graded modules, thus also a natural map

$$\tau : H^1(T_X) \rightarrow \text{Hom}(I/I^2, S)_0.$$

It is a result of Hilbert scheme theory (see e.g. [S] ch.9) that there exists a natural map

$$p_0 : \text{Hom}(I/I^2, S)_0 \rightarrow H^0(N_{X|\mathbb{P}^n}),$$

where $H^0(N_{X|\mathbb{P}^n})$ parametrizes the first-order deformations of X in \mathbb{P}^n . Furthermore, under the hypothesis that X be projectively normal, p_0 is an isomorphism.

By composition, $s := p_0 \circ \tau : H^1(T_X) \rightarrow H^0(N_{X|\mathbb{P}^n})$ is a section of the normal sequence $0 \rightarrow T_X \rightarrow T_{\mathbb{P}^n} \rightarrow N_{X|\mathbb{P}^n} \rightarrow 0$.

3 Curves

Let $\mathcal{X} \rightarrow B$ be a smooth family of curves of genus $g \geq 3$. The sequence (3) of the previous section has the following natural interpretation in terms of moduli, when considered in degree $k = 2$.

Let B be any open subset of the moduli space of curves, outside of the locus of curves with automorphisms. Recall that over B exists the universal family $\psi : \mathcal{C} \rightarrow B$ and the period map $\tau : \mathcal{M}_g \rightarrow \mathcal{A}_g$, where \mathcal{A}_g is the moduli space of principally polarized abelian varieties of dimension g , becomes an embedding when restricted to B .

In such a situation, there are the following identifications

$$\mathcal{I}_2(K_{\mathcal{C}|B}) \simeq N_{\tau(B)|\mathcal{A}_g}^*, \quad \text{Sym}^2(\psi_* K_{\mathcal{C}|B}) \simeq T_{\mathcal{A}_g}^*|_{\tau(B)}, \quad \psi_* K_{\mathcal{C}|B}^2 \simeq T_{\tau}^*(B),$$

hence, dualizing (3), we obtain the normal sequence

$$0 \rightarrow T_{\tau(B)} \rightarrow T_{\mathcal{A}_g}|_{\tau(B)} \rightarrow N_{\tau(B)|\mathcal{A}_g} \rightarrow 0.$$

Thus $\Pi_2 : T_{\tau(B)} \rightarrow \text{Hom}(\mathcal{I}_2(K_{\mathcal{C}|B}), \psi_* K_{\mathcal{C}|B}^2)$ is the same as the 2ff of the sheaf sequence above. Thanks to theorem 2.1, we see that ρ is (a factor of) the 2ff of the (local) embedding given by the period map for curves. Karpishpan [K] defines a 2ff of period maps coming from VHS, and asks the question, whether, in the case of curves, the 2ff lift the second Wahl map. In theorem 3.1 below we give a positive answer to this question, by lifting the Wahl map

to ρ . It should be remarked that in [G] p.37-8, a similar lifting of the Wahl map is constructed. Indeed, Green states the assertion of theorem 3.1 for the case $L = K_C$, referring for the proof to unpublished joint work with Griffiths. For ease of reference, we collect here a few well known facts about Wahl maps and Schiffer variations (see e.g. [W] and [Gr].)

Wahl maps

Let C be a smooth projective curve. Set $S := C \times C$ and let $\Delta \subseteq S$ be the diagonal subset. Given a line bundle L on C , and $\lambda_1, \dots, \lambda_r$ a basis of $H^0(L)$ as before, define $L_S := p_1^*L \otimes p_2^*L$, p_i being the projection on the i -th factor, $p_i : S \rightarrow C$, $i = 1, 2$.

Wahl maps are the natural maps:

$$\mu_n : H^0(S, L_S(-n\Delta)) \rightarrow H^0(S, L_S(-n\Delta)|_\Delta) \simeq H^0(C, L^2 \otimes K_C^n).$$

Especially, $I_2(L)$ can be identified with a subspace of $H^0(S, L_S(-2\Delta))$. We are interested in the restriction of μ_2 to $I_2(L)$, i.e. the map

$$\mu_2 : I_2(L) \rightarrow H^0(L^2 \otimes K_C^2).$$

Also, we recall the local expression of μ_2 . In local coordinates, λ_i is of type $\lambda_i = \phi_i \ell$, with ϕ_i a holomorphic function and ℓ a local generator of L ; then $\sum a_{ij} \lambda_i \otimes \lambda_j$ is an element of $I_2(L)$ iff $\sum a_{ij} \phi_i \phi_j$ is identically zero, and, since the a_{ij} are symmetric, also $\sum a_{ij} \phi_i \phi_j = 0$. The local expression of μ_2 is

$$\mu_2 \left(\sum a_{ij} \lambda_i \otimes \lambda_j \right) = \sum a_{ij} \ddot{\phi}_i \phi_j \ell^2 \otimes dz^2. \quad (5)$$

Schiffer variations

As above, let L be a line bundle over a curve C , $\deg L \geq 2$. For any point $P \in C$ consider the exact sequence $0 \rightarrow L^{-1} \rightarrow L^{-1}(P) \rightarrow L^{-1}(P)|_P \rightarrow 0$. The image of the induced natural map $\delta : H^0(L^{-1}(P)|_P) \rightarrow H^1(L^{-1})$ has dimension one. Every generator of $\text{im } \delta$ is called a Schiffer variation of L at P , denoted ξ_P . It is easy to check that, via the Dolbeault isomorphism $H^1(L^{-1}) = H^{0,1}(L^{-1})$, ξ_P is represented by a form

$$\theta_P = \frac{1}{z} \bar{\partial} b \otimes \ell^*, \quad (6)$$

where z is a holomorphic coordinate on C around P , b is a bump function around P and ℓ^* is the dual of a local generator of L .

A lifting of the Wahl map

Theorem 3.1 *The following diagram*

$$\begin{array}{ccc} I_2(L) & \xrightarrow{\rho} & \mathrm{Sym}^2 H^0(L \otimes K_C) \\ \mu_2 \downarrow & & \downarrow \mathbf{m} \\ H^0(L^2 \otimes K_C^2) & \xlongequal{\quad} & H^0(L^2 \otimes K_C^2) \end{array}$$

is commutative up to a constant.

The strategy of proof is the following.

Given $Q \in I_2(L)$, in order to check $\mu_2(Q) = (\mathbf{m} \circ \rho)(Q)$, it is enough to evaluate both at every point P in some open subset U of X . $\mu_2(Q)(P)$ is easily computed in terms of (5). To evaluate $(\mathbf{m} \circ \rho)(Q)(P)$ we express the dual map \mathbf{m}^* in terms of Schiffer variations: namely, if v_P is the evaluation map at P , then, up to a constant, $\mathbf{m}^*(v_P) = \xi_P \odot \xi_P$. Thus $(\mathbf{m} \circ \rho)(Q)(P) = (\xi_P \odot \xi_P)(\rho(Q))$, and the right hand value is computed making use of the explicit representation (6) of the Schiffer variation at P .

Proof: If $I_2(L) = 0$, there is nothing to prove, so we can suppose $h^0(L) > 1$ and $\deg L \geq 2$. For any $P \in C$, let v_P be the evaluation map at P , defined on $H^0(L^2 \otimes K_C^2)$.

Fix $P_0 \in C$, choose a coordinate z on C and a trivialization of L , with local generator ℓ , around P_0 . Also, let b be a bump function around P_0 , and let $U \subseteq C$ be an open neighborhood of P_0 on which $b \equiv 1$. We can suppose that both the coordinate z on C and the trivialization of L are defined on U .

For all $P \in U$, via the identification $(L^2 \otimes K_C^2)_P \simeq \mathbb{C}$ coming from the chosen trivialization, we can think of v_P as an element of $H^0(L^2 \otimes K_C^2)^*$. We want to express its image, under the dual multiplication map

$$\mathbf{m}^* : H^0(L^2 \otimes K_C^2)^* \rightarrow \mathrm{Sym}^2 H^0(L \otimes K_C)^*,$$

in terms of the Schiffer variation ξ_P of L at P represented by the form $\theta_P = \frac{1}{z-z(P)} \bar{\partial} b \otimes \ell^*$.

Claim $\mathbf{m}^*(v_P) = \frac{1}{(2\pi i)^2} \xi_P \odot \xi_P \in \text{Sym}^2 H^1(L^{-1})$.

Locally around P , $\tau \in \text{Sym}^2 H^0(L \otimes K_C)$ has the form $\tau = \sum a_{ij}(\phi_i \ell \otimes dz) \otimes (\phi_j \ell \otimes dz)$. Thus

$$v_P(\mathbf{m}(\tau)) = \sum a_{ij} \phi_i(P) \phi_j(P).$$

On the other hand,

$$(\xi_P \odot \xi_P)(\tau) = (2\pi i)^2 \sum a_{ij} \phi_i(P) \phi_j(P).$$

Now, let $Q = \sum a_{ij} \lambda_i \otimes \lambda_j \in I_2(L)$; to prove $k \cdot \mu_2(Q) = (\mathbf{m} \circ \rho)(Q)$, k constant, it is enough to show that, for any $P \in U$, $k \cdot \mu_2(Q)(P) = (\mathbf{m} \circ \rho)(Q)(P)$, hence, for some constant h ,

$$h \cdot v_P(\mu_2(Q)) = (\xi_P \odot \xi_P)(\rho(Q)).$$

As before write $\lambda_i = \phi_i \ell$. By (5)

$$v_P(\mu_2(Q)) = \sum a_{ij} \ddot{\phi}_i(P) \phi_j(P)$$

and, by (2)

$$\begin{aligned} (\xi_P \odot \xi_P)(\rho(Q)) &= \xi_P \cdot \rho_Q(\xi_P) \\ &= \left(\frac{1}{z - z(P)} \bar{\partial} b \otimes \ell^* \right) \cdot \sum a_{ij} \lambda_i \partial h_j \\ &= \int_C \frac{1}{z - z(P)} \bar{\partial} b \sum a_{ij} \phi_i \frac{\partial h_j}{\partial z} dz. \end{aligned}$$

Write $\Psi(z) := \sum a_{ij} \phi_i \frac{\partial h_j}{\partial z}$, then the value of the last integral is $2\pi i \Psi(P)$.

To evaluate $\Psi(P)$ we proceed as follows: $\theta_P \lambda_i = \frac{\phi_i}{z - z(P)} \bar{\partial} b$, so, in $C - \{P\}$, we have the equality (cf.(1)) $\gamma_i + \bar{\partial} h_i = \bar{\partial} \left(\frac{b \phi_i}{z - z(P)} \right)$, hence $\gamma_i = \bar{\partial} g_i$, with $g_i = \frac{b \phi_i}{z - z(P)} - h_i$.

Define $\eta_i := \partial g_i$.

Lemma 3.2 (cf. [Pi] 4.8) *The η_i are all proportional, hence*

$$\rho_Q(\xi_P) = \eta \sum a_{ij} b_i \lambda_j \in H^0(L \otimes K_C),$$

where η is a differential of second kind, multiple of the η_i , having only a double pole at P , and $b_i = \frac{\eta_i}{\eta}(P)$ are constants.

Proof: *Step 1* $\sum a_{ij} \lambda_i \partial h_j = -\sum a_{ij} \lambda_i \eta_j$.

In the first place, note that η_i is holomorphic in $C - \{P\}$. Indeed, γ_i is harmonic, thus

$$\bar{\partial} \eta_i = \bar{\partial} \partial g_i = -\partial \bar{\partial} g_i = -\partial \gamma_i = 0.$$

Hence, $\sum a_{ij} \lambda_i \partial(\frac{b\phi_j}{z-z(P)}) = \sum a_{ij} \lambda_i \partial h_j + \sum a_{ij} \lambda_i \eta_j$ is holomorphic on $C - \{P\}$ and, in a neighborhood of P where $b \equiv 1$, it has the form

$$\left(\sum a_{ij} \phi_i \frac{\partial}{\partial z} \left(\frac{\phi_j}{z-z(P)} \right) \right) \ell \otimes dz = \sum a_{ij} \phi_i \left(-\frac{\phi_j}{(z-z(P))^2} + \frac{\dot{\phi}_j}{z-z(P)} \right) \ell \otimes dz = 0,$$

because $\sum a_{ij} \phi_i \dot{\phi}_j = \sum a_{ij} \phi_i \dot{\phi}_j = 0$, so $\sum a_{ij} \lambda_i \partial(\frac{b\phi_j}{z-z(P)})$ is identically zero.

Step 2 $\eta_i \in H^0(K_C(2P))$.

On U , where $b \equiv 1$, η_i has the form

$$\eta_i = \left(-\frac{\phi_i(P)}{(z-z(P))^2} + f_i(z) \right) dz \quad (7)$$

with $f_i(z)$ a holomorphic function, hence η_i is a meromorphic form, with a double pole at P .

Step 3 η_i are proportional.

If C has genus $g = 0$, then $h^0(K_C(2P)) = 1$.

If $g \geq 1$, by definition, $\eta_i + \gamma_i = \partial g_i + \bar{\partial} g_i = dg_i$, so $[\eta_i] = -[\gamma_i] \in H^1(C - \{P\}, \mathbb{C})$, hence $\eta_i \in H^0(K_C(2P)) \cap H^{0,1}(C)$, via the inclusion $H^0(K_C(2P)) \hookrightarrow H^1(C - \{P\}, \mathbb{C}) \simeq H^1(C, \mathbb{C})$.

Now, $\dim H^0(K_C(2P)) \cap H^{0,1}(C) = 1$: it is a consequence of $h^0(K_C(2P)) = g + 1$ and $H^0(K_C) \subseteq H^0(K_C(2P))$. It follows that the η_i are all proportional. ■

To finish the proof of the theorem, take η to be the only differential having local expression $\eta = (-\frac{1}{(z-z(P))^2} + f(z))dz$, $f(z)$ holomorphic. So $\eta_i = \phi_i(P)\eta$, and we see that $\Psi(z)$, given by the local expression of $-\sum a_{ij} \lambda_i \eta_j$, has the form $\Psi(z) = \frac{1}{2} \sum a_{ij} \ddot{\phi}_i(P) \phi_j(P) + o(z-z(P))$. Thus,

$$\Psi(P) = \frac{1}{2} \sum a_{ij} \ddot{\phi}_i(P) \phi_j(P),$$

hence $v_P(\mu_2(Q)) = (\xi_P \odot \xi_P)(\rho(Q))$, up to a constant factor. ■

Remark 3.3 Recall that, if S is a subbundle of the hermitian bundle E , and ∇_E and ∇_S are the metric connections, then the 2ff of the embedding $S \hookrightarrow E$ gives information on the curvature of S , because of the relation

$$\nabla_S = \nabla_E - \Pi.$$

So, the lemma above makes possible explicit computations about the curvature of the moduli space of curves.

4 Generalizations

Pairs of vector bundles

Let E and F be vector bundles on a smooth projective variety X . As in the previous section, define $Y := X \times X$, with $p_i : Y \rightarrow X$, $i = 1, 2$, projections on the i -th factor and $\Delta \subseteq Y$ the diagonal subset. Tensoring the exact sequence $0 \rightarrow \mathcal{I}_\Delta^{n+1} \rightarrow \mathcal{I}_\Delta^n \rightarrow \mathcal{I}_\Delta^n / \mathcal{I}_\Delta^{n+1} \rightarrow 0$ by $E \boxtimes F := p_1^* E \otimes p_2^* F$, and taking cohomology, we get

$$0 \rightarrow H^0(Y, E \boxtimes F \otimes \mathcal{I}_\Delta^{n+1}) \rightarrow H^0(Y, E \boxtimes F \otimes \mathcal{I}_\Delta^n) \rightarrow H^0(X, E \otimes F \otimes \text{Sym}^n \Omega_X^1).$$

Following the notation of [P], we define the k -th module of relations of E and F as $R_k(E, F) := H^0(Y, E \boxtimes F \otimes \mathcal{I}_\Delta^k) \subseteq H^0(Y, E \boxtimes F) \simeq H^0(X, E) \otimes H^0(X, F)$. Note also that, when $E = F$, $I_2(E)$ is a submodule of $R_2(E, E)$.

We now extend the definition of ρ to $R_2(E, F)$, to obtain a map, still denoted by ρ ,

$$\rho : R_2(E, F) \rightarrow \text{Hom}(H^{p,q}(X, E^*), H^{p+1,q-1}(X, F)).$$

Let $\lambda_i, i = 1, \dots, s$, and $\mu_j, j = 1, \dots, t$, be bases of $H^0(X, E)$ and $H^0(X, F)$ respectively. Because of the inclusion $R_2(E, F) \subseteq H^0(X, E) \otimes H^0(X, F)$, an element $P \in R_2(E, F)$ can be written as $P = \sum a_{ij} \lambda_i \otimes \mu_j$. If $\xi \in H^{p,q}(E^*)$ and $\theta \in A^{p,q}(E^*)$ is a Dolbeault representative of ξ , then $\theta \lambda_i \in A^{p,q}(X)$ are $\bar{\partial}$ -closed forms having harmonic decompositons $\theta \lambda_i = \gamma_i + \bar{\partial} h_i$. We define $\rho_P(\xi)$ as the Dolbeault cohomology class of the $(p+1, q-1)$ -form

$$\sigma_P(\theta) := \sum_{ij} a_{ij} \bar{\partial} h_i \otimes \mu_j.$$

Theorem 4.1 *The map*

$$\begin{aligned} \rho : R_2(E, F) &\rightarrow \text{Hom}(H^{p,q}(E^*), H^{p+1,q-1}(F)) \\ P &\rightarrow (\xi \mapsto \rho_P(\xi)) \end{aligned}$$

is well defined and linear.

Proof: The proof runs along the same lines of that of 1.2.

(i) $\sigma_P(\theta)$ is $\bar{\partial}$ -closed.

Clearly, $\bar{\partial}\sigma_P(\theta) = \bar{\partial}\left(\sum a_{ij}\partial h_i \otimes \mu_j\right) = -\sum a_{ij}\partial(\theta\lambda_i) \otimes \mu_j$ as before, so we perform again a local computation.

Let U be an open subset of X on which E and F are both locally trivial, and let $\ell_i, i = 1, \dots, q$, and $m_j, j = 1, \dots, r$, be local basis of E and F respectively, with $\ell_i^*, i = 1, \dots, q$, the dual basis of E^* , then locally $\lambda_i = \sum_l \gamma_{il}\ell_l$ and $\mu_j = \sum_k \beta_{jk}m_k$, with γ_{il}, β_{jk} holomorphic functions. An element $P = \sum a_{ij}\lambda_i \otimes \mu_j \in R_2(E, F)$, being a section of a bundle twisted by \mathcal{I}_Δ^2 , vanishes on X to the second order, which locally translates into $\sum_{ij} a_{ij}\gamma_{il}\beta_{jk} = 0$ and $\sum_{ij} a_{ij}\beta_{jk}\partial\gamma_{il} = 0$ for all l, k . Also, $\theta \in A^{p,q}(E^*)$ on U is of the form $\theta = \sum_l \omega_l \ell_l^*$, with $\omega_l \in A^{p,q}(X)$, hence $\theta\lambda_i = \sum_l \gamma_{il}\omega_l$. It follows that locally

$$\begin{aligned} \sum_{ij} a_{ij}\partial(\theta\lambda_i) \otimes \mu_j &= \sum_{ij} a_{ij}\partial\left(\sum_l \gamma_{il}\omega_l\right) \otimes \sum_k \beta_{jk}m_k \\ &= \sum_{ijkl} a_{ij}\beta_{jk}(\partial\gamma_{il} \wedge \omega_l + \gamma_{il} \wedge \partial\omega_l) \otimes m_k \\ &= \sum_{kl} \left(\sum_{ij} a_{ij}\beta_{jk}\partial\gamma_{il}\right) \wedge \omega_l \otimes m_k + \\ &\quad \sum_{kl} \left(\sum_{ij} a_{ij}\gamma_{il}\beta_{jk}\right) \wedge \partial\omega_l \otimes m_k \\ &= 0 \end{aligned}$$

thus $\sigma_P(\theta)$ is $\bar{\partial}$ -closed.

(ii) $\rho_P(\xi)$ does not depend on the choice of θ .

The argument is completely similar to that of 1.2. ■

Remark 4.2 By Serre duality, the range of the map ρ is $\text{Hom}(H^{p,q}(E^*), H^{p+1,q-1}(F)) = (H^{p,q}(E^*))^* \otimes H^{p+1,q-1}(F) = H^{n-p,n-q}(E) \otimes H^{p+1,q-1}(F)$. Also, it is a consequence of Kunneth formula that $H^{r,s}(X \times X, E \boxtimes F) = \bigoplus_{\substack{i+j=r \\ h+k=s}} H^{i,h}(X, E) \otimes H^{j,k}(X, F)$.

Thus, adding together all the maps ρ , we have a natural map (still denoted by ρ)

$$\rho : H^0(X \times X, E \boxtimes F \otimes \mathcal{I}_\Delta^2) \rightarrow H^{n-1}(X \times X, E \boxtimes F \otimes \Omega_{X \times X}^{n+1}).$$

Harmonic bundles

We collect here some definitions and known facts about Higgs and harmonic bundles (cf. [Si].)

Let X be a compact Kähler manifold of dimension n , with Kähler form ω . A Higgs field (or Higgs bundle) is a pair (E, ϕ) , with E a holomorphic vector bundle and $\phi : E \rightarrow E \otimes \Omega_X^1$ a holomorphic map such that $\phi \wedge \phi = 0$. Associated to ϕ there is the operator $D'' := \bar{\partial} + \phi : \mathcal{A}^0(E) \rightarrow \mathcal{A}^1(E)$, with

$$D''(f\sigma) = \bar{\partial}f \cdot \sigma + fD''\sigma, \quad D''^2 = 0.$$

The Dolbeault cohomology $H_{Dolb}^*(E)$ is defined as the hypercohomology of the complex

$$E \otimes \Omega_X^\bullet \xrightarrow{\phi} E \otimes \Omega_X^1 \xrightarrow{\phi} E \otimes \Omega_X^2 \xrightarrow{\phi} \dots$$

$H_{Dolb}^*(E)$ is isomorphic to the cohomology of the complex $\mathcal{A}^i(E) \xrightarrow{D''} \mathcal{A}^{i+1}(E)$. Note that D'' defines a different holomorphic structure on E . If E is endowed with a hermitian metric H , define D'_H as the operator for which $D = D'_H + D''$ is the hermitian connection, with respect to the holomorphic structure of E associated to D'' . When D is flat, E is called a harmonic bundle (and H is a harmonic metric.)

A fundamental result in the theory of Higgs and harmonic bundles is the following.

Theorem 4.3 (cf. [Si] theorem 1)

A Higgs bundle has a harmonic metric if and only if it is polystable (i.e. direct sum of stable Higgs bundles having the same slope) and $c_1(E)[\omega]^{n-1} = c_2(E)[\omega]^{n-2} = 0$.

Conversely, a flat bundle (with a metric) comes from a Higgs bundle if and only if it is semisimple.

The following hold for harmonic bundles:

- (i) The Kähler identities.
- (ii) The associated harmonic decomposition

$$\begin{aligned}\mathcal{A}^\bullet(E) &= \mathcal{H}^\bullet(E) \oplus \text{im } D \oplus \text{im } D^* \\ &= \mathcal{H}^\bullet(E) \oplus \text{im } D'' \oplus \text{im } D''^*,\end{aligned}$$

$\mathcal{H}(E)$ being the kernel of the laplacian operator $\Delta = DD^* + D^*D = 2(D''D''^* + D''^*D'')$.

- (iii) The principle of two types

$$\ker(D'_H) \cap \ker(D'') \cap (\text{im}(D'_H) + \text{im}(D'')) = \text{im}(D'_H D'').$$

These properties are all one needs to generalize the construction of the map ρ . Let (E, H) be a harmonic bundle, with $D'' = \bar{\partial} + \phi$. Then, for any line bundle L , D'' defines structures of Higgs bundles on both $E \otimes L$ and $E \otimes L^{-1}$, with associated cohomology $H_{Dolb}^*(E \otimes L)$ and $H_{Dolb}^*(E \otimes L^{-1})$.

Theorem 4.4 *Let (E, H) be a harmonic vector bundle on a compact Kähler manifold X , and let L be any line bundle on X . Then is well-defined the map*

$$\begin{aligned}I_2(L) \otimes H_{Dolb}^k(E \otimes L^{-1}) &\rightarrow H_{Dolb}^k(E \otimes L) \\ (Q, \alpha) &\rightarrow [\sum a_{ij} \lambda_i D'_H h_j]\end{aligned}$$

where:

- (i) $\{\lambda_i\}$ is any basis of $H^0(L)$, so that Q can be written as $Q = \sum a_{ij} \lambda_i \otimes \lambda_j$, and
- (ii) h_j is given by the harmonic decomposition $\lambda_j \tilde{\alpha} = \gamma_j + D'' h_j$, $\tilde{\alpha}$ being a form representing α , i.e. $\alpha = [\tilde{\alpha}]$, with $\tilde{\alpha} \in \mathcal{A}^i(E \otimes L^{-1})$ and $D'' \tilde{\alpha} = 0$.

Proof: The proof is completely analogous to that of proposition-definition 1.2. ■

The simplest case is that of a polystable vector bundle with $c_1(E)[\omega]^{n-1} = c_2(E)[\omega]^{n-2} = 0$ —in other terms, i.e. $D'' = \bar{\partial}$ and $H_{Dolb}^k(E) = \oplus_{p=0}^k H^{k-p}(E \otimes \Omega_X^p)$, also ρ is a map

$$I_2(L) \rightarrow \oplus_p H^{n-k+p}(E^* \otimes L \otimes \Omega_X^{n-p}) \otimes H^{k-p-1}(E \otimes L \otimes \Omega_X^{p+1}).$$

Especially, for any degree zero line bundle M on a smooth curve C , there exists a harmonic metric H on M , with metric connection D_H that decomposes as $D_H = D'_H + \bar{\partial}$. Thus we have the map

$$I_2(L) \xrightarrow{\rho} H^0(M^{-1} \otimes L \otimes K_C) \otimes H^0(M \otimes L \otimes K_C).$$

By means of the multiplication map

$$H^0(M^{-1} \otimes L \otimes K_C) \otimes H^0(M \otimes L \otimes K_C) \xrightarrow{m} H^0(L^2 \otimes K_C^2),$$

we have the following generalization of theorem 3.1.

Theorem 4.5 *The diagram*

$$\begin{array}{ccc} I_2(L) & \xrightarrow{\rho} & H^0(M^{-1} \otimes L \otimes K_C) \otimes H^0(M \otimes L \otimes K_C) \\ \mu_2 \downarrow & & \downarrow m \\ H^0(L^2 \otimes K_C^2) & \xlongequal{\quad} & H^0(L^2 \otimes K_C^2) \end{array}$$

is commutative up to a constant.

Proof: The proof, that uses the operators D , D'_H and $\bar{\partial}$ in the rôles of d , ∂ and $\bar{\partial}$ respectively, is analogous to that of theorem 3.1 but for the details noted below.

- (i) We suppose that on U there exists also a trivialization of M , with local generator ν . Then, in the claim, the Schiffer variation $\xi_P \in H^1(M \otimes L^{-1})$ is the one represented by the form $\theta_P = \frac{1}{z-z(P)} \bar{\partial} b \otimes \nu \otimes \ell^*$.
- (ii) The metric is represented on U by a scalar function, still denoted by H . Hence $D'_H : \mathcal{A}^0(M) \rightarrow \mathcal{A}^{0,1}(M)$ locally is

$$D'_H(f\nu) = \left(\partial f + f \frac{\partial H}{H} \right) \otimes \nu.$$

Writing $h_i = l_i \nu$, then $\rho_Q(\xi)$ is represented by the form $\Psi(z) \nu \otimes \ell \otimes dz$, with

$$\Psi(z) = \sum a_{ij} \phi_i \left(\frac{\partial l_j}{\partial z} + l_j \frac{\partial \log H}{\partial z} \right).$$

- (iii) Steps 1 and 2 of lemma 3.2 carry through the present situation, with the local expression (7) for the form η_i now becoming

$$\eta_i = \left(-\frac{\phi_i(P)}{(z-z(P))^2} + \frac{\partial \log H}{\partial z}(z(P)) \frac{\phi_i(P)}{z-z(P)} + f_i(z) \right) \nu \otimes dz. \quad (8)$$

To prove step 3, we argue as follows.

Assume that P is not a base point for L , i.e. not all $\phi_i(P) = 0$. As a consequence of (8), we have that $\phi_j(P)\eta_i - \phi_i(P)\eta_j$ is a D'_H -harmonic form defined on all C . Now $\eta_i + \gamma_i = D'_H g_i + \bar{\partial}g_i = D_H g_i$, so

$$\phi_j(P)\eta_i - \phi_i(P)\eta_j + \phi_j(P)\gamma_i - \phi_i(P)\gamma_j = D_H \left(\phi_j(P)g_i - \phi_i(P)g_j \right).$$

The equality above shows that the D_H -harmonic form on the left hand side is D_H -exact, hence it is zero, because of the harmonic decomposition. Especially, its D'_H -part is zero, hence $\phi_j(P)\eta_i = \phi_i(P)\eta_j$. So the η_i 's are proportional and the final computation of the proof can still be performed. ■

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e-mail addresses: `colombo@mat.unimi.it`
`pirola@dimat.unipv.it`
`tortora@mat.unimi.it`